

Vortex mechanics in planar nano-magnets

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(Dated: February 5, 2013)

Collective-variable approach for study of non-linear dynamics of magnetic textures in planar nano-magnets is proposed. The variables are just arbitrary parameters (complex or real) in the specified analytical function of complex variable, describing the texture in motion. Starting with such a function, a formal procedure is outlined, allowing to obtain the (non-linear) system of differential equations of motion for the variables. The resulting equations are equivalent to Landau-Lifshitz-Gilbert dynamics as much as definition of collective variables allows it. Apart from collective-variable specification, the procedure does not involve any additional assumptions (such as translational invariance or steady-state motion). As an example, equations of weakly non-linear motion of magnetic vortex are derived and solved analytically. A simple formula for dependence of vortex precession frequency on its amplitude is derived. The results are verified against special cases from the literature and agree quantitatively with experiments and simulations.

PACS numbers: 75.60.Ch, 75.70.Kw, 85.70.Kh

Keywords: magnetization dynamics, magnetic nano-dots, magnetic vortex

Unlike bulk magnets or thin films (whose magnetic texture consists of domains and domain walls, extensively studied during the last century in the framework of micromagnetics) nano-magnets, due to their restricted geometry, display markedly different textures, consisting of vortices and anti-vortices, some of which can be bound to the surface[1–3]. This opens up a wealth of possibilities for new physics, not studied in the established science of micromagnetics. Some micromagnetic approaches (such as Thiele equation[4] for steady-state uniform translation of magnetic texture) can be successfully applied to study of motion of single magnetic vortex in bigger nano-magnets[5, 6], but the full account for restricted geometry (their most essential feature) and multi-vortex dynamics requires a different formalism, which is free from assumptions, specific to micromagnetics.

Nevertheless, Thiele equation is still the most useful analytical tool for study of nano-magnet dynamics. Another approach, based on volume averaging of the Landau-Lifshitz-Gilbert (LLG) equation in vector form, produces qualitative agreement with micromagnetic simulations and the experiment[7, 8]. But it leads to underestimate of texture mobility[8] and vortex precession frequency[7], because LLG equation is non-linear (due to constraint on the length of magnetization vector, masked when it is written in vector form) and it's volume averaging (a form of linear superposition) is, generally, not justified. Consideration of spin-waves on magnetic vortex background[9, 10] does reproduce translation of magnetic vortex and predicts higher-energy spin-wave modes, but is limited to linear consideration of small vortex displacements only. Including higher order terms in deviation of magnetization from (magnetic vortex) background is not only mathematically hard, but also bound to have difficulties reproducing the complicated non-linear motion of the multi-vortex texture, which may completely depart from the original static background. Non-uniform

background also makes it difficult to deal with non-local dipolar forces, which is the reason why in many of such works (Refs 9 and 10 in particular) dipolar interaction is replaced by local in-plane anisotropy, making the equations partial differential (instead of integral partial differential). This is justified in the limit of vanishing nano-element's thickness, but quantitative agreement with experiments and simulations in wide range of geometries is only possible when magnetostatic interaction is fully accounted for.

Here another approach to magnetization dynamics in planar nano-dots is proposed. It is a collective-variable theory, capable of dealing with complex multi-vortex configurations (fully describing relative motion and deformation of constituent vortices). Its only approximation lies in the definition of collective variables, which are just an arbitrary parameters in a complex function of complex variable. Given such a function, the approach produces a system of ordinary differential equations (ODEs) of motion (with no integral terms even in the presence of magnetostatic interaction) for these variables. They are exactly equivalent to Landau-Lifshitz-Gilbert equation, apart from the restriction, implied by the collective variable definition. The approach assumes neither translational invariance, nor steady state motion and is fully capable of describing non-linear dynamics (if one can solve the resulting non-linear ODEs). The external field, spin-torque and other potential energy terms can be easily added without sacrificing simplicity (dynamical equations still remain ODEs). It also allows to include phenomenological dissipation, akin to Gilbert's damping term in LLG. As an illustration, the equations of motion for linear and weakly non-linear magnetic vortex dynamics in circular nano-dots are derived and solved. As a check, the well known analytical result for vortex precession frequency (originally obtained by solving the Thiele equation) is then recovered in the limiting case of large

dots.

The original motivation for this work comes from recently published description of low-energy (single- and multi-vortex) magnetic configurations in planar nanodots of arbitrary shape in terms of functions of complex variable[3]. The present approach can be thought as a way to “animate” these configurations by making them move in accordance with LLG dynamics. Despite this, one may easily generalize it to other sets of trial functions without substantial sacrifice in simplicity.

The usual starting point for consideration of magnetization dynamics is LLG equation[11]. It is very well suited for numerical computations, but is not good for analytical ones. This is because the non-uniform effective field in LLG, around which magnetization vectors precess, depends, in turn, on the whole magnetization vector field (if dipolar interaction is properly taken into account). This makes it non-linear integral partial differential equation, which is extremely hard to solve analytically. Therefore, instead of LLG, let us go deeper and consider, as a starting point, kinetic Lagrangian density, which was first introduced by Döring[12]:

$$\tau = -\frac{M_S}{\gamma} (\cos \theta - \cos \theta_0) \frac{\partial \varphi}{\partial t}, \quad (1)$$

where θ and φ are polar and azimuthal angle of the magnetic moment \vec{M} in a spherical coordinate system, t is time, $\gamma \simeq 1.76 \cdot 10^{11} \text{rad}/(\text{s T})$ is gyromagnetic ratio, M_S is saturation magnetization and θ_0 is a constant. Parametrization of \vec{M} via spherical angles conveniently satisfies the constraint $|\vec{M}| = M_S$, leaving only two of its components independent (and bounded). The system of two Euler-Lagrange equations for the extremum of the corresponding action over θ and φ with additional time-independent potential energy terms subtracted, are equivalent to Landau-Lifshitz equation[12, 13]. They do not depend on θ_0 , which can be used to ensure that τ is zero at the boundary of the magnet.

Collective variable approach is then similar to the Ritz method[14] of solving boundary value problems: first, one selects a trial function, parametrizing a wide set of possible solutions, and then finds the values of parameters, giving the correct answer (extremalizing a certain functional, as per variational principle). Ritz method in its original formulation finds wide applications in micro-magnetics for solving static problems. Dynamics is not much different. One may look for the extremum of action of the full Lagrangian, including kinetic and potential energy, parametrized by a certain set of scalar parameters. Condition for this extremum produces dynamical equations for the parameters, allowing to compute their evolution in time.

While the above general recipe is applicable to arbitrary choice of trial functions, to make further consideration more specific, let us focus on a particular very

general family[3]. Consider a cylinder, made of soft ferromagnetic material, placed in the origin of the Cartesian coordinate system in such a way that the axis Z is perpendicular to the cylinder face \mathcal{D} , which is not necessarily circular. distribution of magnetization vector $\vec{M}(\vec{r})$, $\vec{r} = \{X, Y, Z\}$ inside can be assumed to be independent on the coordinate Z . It can be conveniently parametrized by a complex function $w(z, \bar{z})$ of complex variable $z = X + iY$ (line on top denotes the complex conjugation, so that $\bar{z} = X - iY$), and to express normalized magnetization $\vec{m} = \vec{M}/M_S$ as

$$m_X + im_Y = \frac{2w}{1 + w\bar{w}} \quad (2)$$

$$m_Z = \pm \frac{1 - w\bar{w}}{1 + w\bar{w}}, \quad (3)$$

which automatically satisfies the constraint $|\vec{m}| = 1$. The sign in m_Z controls polarization of the vortex core: $m_Z = \pm 1$ at the vortex center.

In flat cylinder the equilibrium static magnetization distributions can be represented as a combination of soliton and meron[3]

$$w(z, \bar{z}) = \begin{cases} f(z)/c_1 & |f(z)| \leq c_1 \\ f(z)/\sqrt{f(z)\bar{f}(\bar{z})} & c_1 < |f(z)| \leq c_2 \\ f(z)/c_2 & |f(z)| > c_2, \end{cases} \quad (4)$$

where $f(z)$ is an analytic ($\partial f(z)/\partial \bar{z} = 0$) function of complex variable z and c_1, c_2 are real constants.

Besides the complex variable z the function $w(z, \bar{z})$ usually depends on a number of other variables like c_1, c_2 , and others hidden inside $f(z)$. The simplest case of a single magnetic vortex (Usov’s anzats[1]) corresponds to

$$f_U(z) = \frac{i(z - A)}{r_V}, \quad (5)$$

with $c_1 = 1$ (absorbed into vortex core radius r_V) and $c_2 \rightarrow \infty$ in (4), since there are no anti-vortices. This magnetization distribution depends on real parameter r_V and complex parameter A , which are the collective variables in this case. The parameter $A = a_X + ia_Y$ is the position of the vortex center. The problem, considered below, is how to find dynamical equations for these (and other similar) collective variables, assuming they are $r_V = r_V(t)$, $A = A(t)$ functions of time t .

The Landau-Lifshitz equation can also be written directly in complex notation[15]. However, as discussed earlier, our starting point will be the kinetic part of the magnetic Lagrangian density (1), expressed through the collective variables and their time derivatives. In the complex notation its ingredients are

$$\cos \theta = \pm \frac{1 - w\bar{w}}{1 + w\bar{w}} \quad (6)$$

$$\exp(i\varphi) = \sqrt{\frac{w}{\bar{w}}} \quad (7)$$

$$\frac{\partial \varphi}{\partial t} = -\frac{i}{2} \frac{\partial}{\partial t} \log \frac{w}{\bar{w}}, \quad (8)$$

so that (1) can be rewritten as

$$\tau = \mp \frac{M_S}{\gamma} \text{Im} \frac{1 - w\bar{w}}{1 + w\bar{w}} \frac{\dot{w}}{w}, \quad (9)$$

where dot over the variable denotes the time derivative and $w(z, \bar{z}, t)$ from (4) depends on time t via the collective variables in the trial function $f(z)$. The meron part of (4), as per our selection of $\theta_0 = \pi/2$, gives no contribution to the kinetic Lagrangian, because $|w(z, \bar{z})| = 1$ inside it, and therefore $\tau \sim m_Z \sim \cos \theta = 0$.

To derive dynamical equations total Lagrangian is needed, which is the Lagrangian density, integrated over the particle volume

$$T = \int_{\mathcal{D} \times L_Z} \tau d^3 \vec{r} = L_Z \int_{\mathcal{D}_S} \tau d^2 \vec{r}, \quad (10)$$

where L_Z is cylinder's thickness and \mathcal{D}_S is part of cylinder's face, occupied by soliton (4), for which $|f(z)| < c_1$ or $|f(z)| > c_2$. It can be simplified for arbitrary f in (4) by noting that

$$\frac{\partial}{\partial t} \log \frac{4w(t)\bar{w}(t')}{(1 + w(t)\bar{w}(t'))^2} = \frac{1 - w(t)\bar{w}(t')}{1 + w(t)\bar{w}(t')} \frac{1}{w(t)} \frac{\partial w(t)}{\partial t} \quad (11)$$

where the variable t' is considered independent and does not take part in differentiation. Interchanging the Im operation and time derivative with the area integral (which is possible because the area element is real and integration is a linear operation) we arrive at the following expression for the total kinetic Lagrangian

$$T = \mp \frac{M_S L_Z}{\gamma} \text{Im} \frac{\partial}{\partial t} \int_{\mathcal{D}'_S} \log \frac{4w(z, t)\bar{w}(\bar{z}, t')}{(1 + w(z, t)\bar{w}(\bar{z}, t'))^2} d^2 z \Big|_{t' \rightarrow t}, \quad (12)$$

where for the purpose of calculating the area integral and differentiating it is assumed that all the collective variables inside \bar{w} and definition of integration region \mathcal{D}'_S (note the prime) depend on the new independent time variable t' , whereas inside w they still depend on t (only after differentiation the t' is replaced by t). It was also noted that inside the soliton \mathcal{D}_S the function w is analytic and does not depend on \bar{z} , while its conjugate \bar{w} does not depend on z . This formula can be checked directly. It can be further simplified if there are no boundary-bound vortices and anti-vortices. In this case it is possible to integrate (12) by parts, making use of Greene's formula

$$\frac{1}{2i} \oint_{\partial \mathcal{D}} u(z, \bar{z}) dz = \iint_{\mathcal{D}} \frac{\partial u(z, \bar{z})}{\partial \bar{z}} d^2 z, \quad (13)$$

for any reasonably good function u , which yields

$$T = \pm \frac{M_S L_Z}{\gamma} \left[\text{Im} \frac{\partial}{\partial t} \int_{\mathcal{D}'_S} \bar{z} \frac{1 - w\bar{w}}{1 + w\bar{w}} \frac{1}{\bar{w}} \frac{\partial \bar{w}}{\partial \bar{z}} d^2 z \right]_{t' \rightarrow t}, \quad (14)$$

where function arguments are omitted, but it is still assumed that all the collective variables inside w depend on t , but all of them inside \bar{w} and definition of the region \mathcal{D}'_S depend on t' .

The expression (12) (and (14) for the case of solitons fully contained inside the particle) is the main result of this paper. To derive equations of motion for collective variables $x_i(t)$, entering the trial function $f(\{x_i\}, z)$, it is now sufficient to write the full Lagrangian:

$$L(\{\dot{x}_i\}, \{x_i\}) = T(\{\dot{x}_i\}, \{x_i\}) - U(\{x_i\}), \quad (15)$$

where U is the potential energy (including exchange, magnetostatic and, possibly, other energy terms); and use it to derive the system of Euler-Lagrange equations

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{x}_i} - \frac{\partial L}{\partial x_i} = 0, \quad (16)$$

which extremize the corresponding action. This allows to treat problems of (multi-) vortex dynamics as problems of classical mechanics. It is also worth noting that apart from restrictions, implied by a particular choice of f , defining collective variables x_i , the above consideration involves no approximations and corresponds to the solution of LLG equation exactly.

Let us now proceed with examples.

First, to illustrate that, similarly to classical mechanics, this theory may suffer from poor selection of trial functions, consider dynamics of a uniformly displaced magnetic vortex (5) with $r_V = \text{const}$ and $A = A(t)$. One may readily check that all three expressions for kinetic Lagrangian (10), (12) and (14) yield 0 in this case. It means that the Euler-Lagrange equations (16) reduce to conditions of static equilibrium, or (in the case of potential energy, consisting of the exchange and magnetostatic terms) that such undeformed vortex stays in the center of the cylinder. This is similar to the conclusion from micromagnetics that moving walls always have their profile different from that of stationary ones[16]. In some sense, one may say that modification of magnetic texture (of domain wall or a magnetic vortex) is the way how it “remembers” that it is moving.

Let us now turn to a more complex trial function, describing the vortex displacement without formation of magnetic charges on cylinder's side[2], which is particular case of a more general class of trial functions[3]

$$f(z) = i \frac{z - (A + \bar{A}z^2)}{r_V}, \quad (17)$$

where again $r_V = \text{const}$ and $A = A(t) = a_X(t) + ia_Y(t)$, $|A| < 1/4$ is a complex pair of collective coordinates. The vortex center, where $f(z_C) = 0$ is located not exactly at $z_C = A$, as in the case of uniformly displaced vortex, but rather at $z_C = (1 - \sqrt{1 - 4A\bar{A}})/(2\bar{A})$. While it is possible to change variables and write the equations of

motion for $z_C(t)$ directly, let us illustrate one of the powers of the present approach, which is a great freedom in selecting the parametrization, and write them for $A(t)$. Also note that here the parameter a in the original expression of Ref. 2 is substituted by $-iA$, which changes nothing but makes the vortex center displacements coincide in phase with the complex parameter A . That is, real A correspond now to real z_C . This again is a matter of convenience and does not change anything, since the parametrization can be arbitrary. The total Lagrangian up to the second order in $|A|$ from (14) and (15) with full account for vortex core shape deformation is

$$\mathcal{L} = \pm \kappa_2 (a_X(t)a'_Y(t) - a_Y(t)a'_X(t)) - k_2 (a_X^2(t) + a_Y^2(t)), \quad (18)$$

where $\mathcal{L} = L/(\mu_0 M_S^2 \pi L_Z R^2)$, k_2 are dimensionless and $\kappa_2 = (1 + r_V^4(4 \log 2 - 3))/(\gamma \mu_0 M_S)$ has units of seconds; k_2 is second order expansion coefficient of the potential energy (which includes exchange and dipolar interaction). The equations of motion (16) are

$$\begin{aligned} \kappa_2 a'_X(t) \pm k_2 a_Y(t) &= 0 \\ \kappa_2 a'_Y(t) \mp k_2 a_X(t) &= 0 \end{aligned} \quad (19)$$

which, for initial conditions $a_X(0) = a_0$, $a_Y(y) = 0$, have the following solution

$$\begin{aligned} a_X(t) &= a_0 \cos(\omega t) \\ a_Y(t) &= \pm a_0 \sin(\omega t), \end{aligned} \quad (20)$$

corresponding to the circular motion of the vortex around the dot center with frequency $\omega = \omega_0 = k_2/\kappa_2$. It is important to note that the direction of vortex motion is not arbitrary. It depends on its core polarization, but not chirality, since T and U are independent on the sign of w or f . The vortices with $m_Z = -1$ at the center rotate clockwise, while the vortices with $m_Z = 1$ counterclockwise. This is in full agreement with simulations and experiments of Ref. 17, but in disagreement with its conclusions, since vortex chirality (included in “handedness”) plays no role in determining the direction of vortex rotation. This also allows to guess that the vortex core polarization in the simulation of Ref. 7 was positive, which is natural to assume, but was not specified by the authors. A similar polarization of the core can be guessed from Fig. 2 in Ref. 6, but with significant uncertainty, since it is masked by low resolution of the measurement in Y direction, as discussed therein.

To make a more rigorous quantitative confirmation of the present theory, let us compute rotation frequency of magnetic vortex in the limit of large flat circular dots with $L_Z \ll R$ and $R \gg L_E$, where $L_E = \sqrt{C/\mu_0 M_S^2}$ is the exchange length, C is the exchange stiffness, and R is dot's radius. In this case $r_V \ll 1$, and $\kappa_2 \simeq 1/(\gamma \mu_0 M_S)$. The second order expansion of the energy of the vortex (17) with vortex core neglected, was first published in Ref. 18. If the exchange contribution of the order

$L_E/R \ll 1$ is neglected in that expansion, the coefficient k_2 in large dots is fully determined by the energy of volume magnetic charges [18]. Converting to SI units, for precession frequency we get

$$\begin{aligned} \nu &= \frac{k_2}{2\pi\kappa_2} = \frac{\gamma\mu_0 M_S}{\pi} \int_0^\infty \frac{f_{MS}(kg)}{k} \left[\int_0^1 r J_1(kr) dr \right]^2 dk \quad (21) \\ \nu &\simeq \gamma\mu_0 M_S g \frac{2(2G-1)}{6\pi^2} \end{aligned} \quad (22)$$

where $f_{MS}(x) = 1 - (1 - e^{-x})/x$, $g = L_Z/R$ and $G \simeq 0.915966$ is Catalan's constant. This expression (apart from measurement units and the value of numerical constant in (22), which is exact here) coincides with the expression for vortex frequency, obtained in Ref. 6 on the basis of Thiele equation and quantitatively confirmed there by experiments on large dots.

Real magnets also inevitably dissipate energy of moving spins in the form of heat. But Lagrangian formalism in its pure form does not include dissipation. It is added externally via a Rayleigh dissipation function D

$$D = \frac{1}{2} \sum_i \sum_j D_{ij} \dot{x}_i \dot{x}_j, \quad (23)$$

which is then included into the right hand side of the Euler-Lagrange equations (16) via an additional term $-\partial D/\partial \dot{x}_i$. The matrix D_{ij} consists of phenomenological dissipation coefficients. Judging from the abstract of the unpublished report by Gilbert[19] it is possible to speculate that Lagrangian formalism was also his starting point and his dissipative term (whose full microscopic justification is still an open problem[20]) has similar origins. Thus, D_{ij} must be related to Gilbert's phenomenological dissipation constant. This relation is, probably, best established by considering the energy balance in the system, but let us leave it for now as an open problem and treat D_{ij} as independent phenomenological parameters. A choice of $D = d(a'_X(t)^2 + a'_Y(t)^2)$ changes the solution (20) into

$$\begin{aligned} a_X(t) &= a_0 e^{-dk_2 t/(d^2 + \kappa_2^2)} \cos(k_2 t \kappa_2/(d^2 + \kappa_2^2)) \\ a_Y(t) &= \pm a_0 e^{-dk_2 t/(d^2 + \kappa_2^2)} \sin(k_2 t \kappa_2/(d^2 + \kappa_2^2)). \end{aligned} \quad (24)$$

As one can see, similarly to the case of linear oscillator, the vortex precession frequency starts to depend (slightly) on (small) damping coefficient.

Finally, let us consider weakly non-linear vortex dynamics by taking into account kinetic and potential energy terms, corresponding to the 4-th order in $|A|$. Continuing the expansion of kinetic Lagrangian (14) with the trial function (17) leads to the following expression

$$\begin{aligned} \mathcal{L} &= \pm (\kappa_2 + \kappa_4 (a_X^2(t) + a_Y^2(t))) (a_X(t)a'_Y(t) - a_Y(t)a'_X(t)) \\ &\quad - k_2 (a_X^2(t) + a_Y^2(t)) - k_4 (a_X^2(t) + a_Y^2(t))^2, \end{aligned} \quad (25)$$

where $\kappa_4 = 2 - r_V^2(23 + r_V^2((6061 - 6397r_V^2)/8 - 1152(1 - r_V^2)\log 2))/(\gamma\mu_0 M_S)$ like κ_2 has units of seconds and k_4 is the next potential energy expansion coefficient. The corresponding equations of motion become non-linear, but they are solved exactly by (20) with

$$\omega = \frac{k_2 + 2a_0^2 k_4}{\kappa_2 + 2a_0^2 \kappa_4} \simeq \omega_0 + 2 \frac{k_4 \kappa_2 - k_2 \kappa_4}{\kappa_2^2} a_0^2 + O(a_0^4). \quad (26)$$

Similarly to other non-linear oscillators, the vortex rotation frequency becomes dependent on the rotation amplitude. Derivation of the expressions for k_2 and k_4 in general case with full account for vortex core deformation is rather cumbersome and, together with the analysis of their dependence on the dot dimensions, will be the subject of another forthcoming paper. Nevertheless, preliminary versions of these expressions are attached in the form of Mathematica file as a supplemental material. They can be used to compute vortex precession frequencies for various dot geometries, not covered here.

Concluding, several equivalent alternative expressions for kinetic Lagrangian (9), (10), (12),(14) of an arbitrary trial function f , defining collective variables in (possibly multi-vortex) magnetic texture[3] in flat nano-elements are derived. They allow to obtain non-linear equations of motion for these variables similarly to the ones in classical Lagrangian mechanics. Apart from collective variable definition, this theory is exact and involves no additional approximations beyond those of Landau-Lifshitz-Gilbert equation. It is validated here by considering magnetic vortex precession in cylindrical nano-dot. In the limit of large flat dots its frequency coincides with experimental data and known theoretical estimations, based on Thiele equation[6]. The question of the direction of vortex rotation is elucidated. It is found to depend on vortex core polarization only and not on its chirality. Also, the analytical solutions for vortex rotation in dissipative magnet are derived, its frequency (24) is dependent on phenomenological damping parameter. Finally, weakly non-linear rotation of vortex is considered, allowing to establish the relation (26) between its frequency and am-

plitude via potential energy expansion coefficients. Expressions for kinetic Lagrangian in (18),(25) for the trial function (17) can be reused in other calculations, including time-dependent external field, spin-torque and other potential energy terms. One may expect them to be as simple as the examples above.

I'd like to thank Vladimir N. Krivoruchko for reading the manuscript and many valuable suggestions.

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